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FAST TRACK COMMUNICATION

Solvable nonlinear evolution PDEs in multidimensional space involving elliptic functions

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Abstract

A *solvable* nonlinear (system of) evolution PDEs in *multidimensional* space, involving *elliptic* functions, is identified, and certain of its solutions are exhibited. An *isochronous* version of this (system of) evolution PDEs in *multidimensional* space is also reported.

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1. Introduction

Recently certain *nonlinear* (systems of) evolution PDEs have been investigated [1], whose *solvability* in *multidimensional* space had been pointed out over a dozen years ago [2]. (*Terminology*: a *nonlinear* evolution PDE is considered *solvable* if its solution can be reduced to solving *linear* PDEs (themselves generally solvable by standard techniques, such as the Fourier transform and quadratures) and possibly also to solving (explicitly known) algebraic or transcendental equations, including the inversion of (explicitly known) transformations—but without having to solve any *nonlinear differential* equation). These results have been extended quite recently [3] by identifying a (multiparametric) *solvable* (system of) nonlinear evolution PDEs involving *trigonometric* (or *hyperbolic*) functions. Purpose and scope of the present paper is to extend these results to a class of nonlinear evolution PDEs in multidimensional space involving *elliptic* functions via an approach analogous to that of [4].

The (system of) nonlinear evolution PDEs treated in this paper read as follows:

$$\alpha\psi_{n,t} + \beta\psi_{n,tt} + \eta\Delta\psi_n = \beta\theta_t\psi_{n,t} + \eta(\vec{\nabla}\theta)\cdot(\vec{\nabla}\psi_n) + 2\sum_{m=1,m\neq n}^{N}\zeta(\psi_n-\psi_m)[\beta\psi_{n,t}\psi_{m,t} + \eta(\vec{\nabla}\psi_n)\cdot(\vec{\nabla}\psi_m)], \qquad (1a)$$

$$\alpha\theta_t + \beta\left(\theta_{tt} - \frac{1}{2}\theta_t^2\right) + \eta\left[\Delta\theta - \frac{1}{2}(\vec{\nabla}\theta)\cdot(\vec{\nabla}\theta)\right]$$
$$= 2f - 2\sum_{m,n=1}^{N}\gamma(\psi_n - \psi_m)[\beta\psi_{n,t}\psi_{m,t} + \eta(\vec{\nabla}\psi_n)\cdot(\vec{\nabla}\psi_m)]. \quad (1b)$$

Notation: N is a positive integer, $N \geqslant 2$; indices such as n, m run from 1 to N unless otherwise indicated; the $N+1$ dependent variables are $\theta \equiv \theta(\vec{r}, t)$ and $\psi_n \equiv \psi_n(\vec{r}, t)$ (but the latter with a constraint, see below); the independent variable t denotes the *time*, and when subscripted indicates partial differentiation (e.g., $\theta_t \equiv \partial\theta/\partial t$, $\psi_{n,tt} \equiv \partial^2\psi_n/\partial t^2$); the independent (*space*) variable $\vec{r}$ is an S-vector, $\vec{r} \equiv (r_1, \ldots, r_S)$, with S being an *arbitrary* positive integer; $\vec{\nabla}$ respectively $\Delta \equiv \vec{\nabla}\cdot\vec{\nabla}$ denote the standard gradient respectively Laplacian operator in S-dimensional space, $\vec{\nabla} \equiv (\partial/\partial r_1, \ldots, \partial/\partial r_S)$, $\Delta \equiv \sum_{s=1}^{S}\partial^2\big/\partial r_s^2$; a dot sandwiched among two S-vectors denotes the standard scalar product, $\vec{u}\cdot\vec{v} \equiv \sum_{s=1}^{S}(u_s v_s)$; α, β, η and f are *arbitrary* constants—although, as it will be clear from the following, they might also depend on the time or space variables without spoiling the solvable character of this system of PDEs; and the functions $\zeta(z) \equiv \zeta(z|\omega, \omega')$ and $\gamma(z) \equiv \gamma(z|\omega, \omega')$ are related to the Weierstrass sigma function $\sigma(z|\omega, \omega')$ by the standard relations

$$\zeta(z|\omega,\omega') = \frac{\sigma_z(z|\omega,\omega')}{\sigma(z|\omega,\omega')}, \qquad \gamma(z|\omega,\omega') = \frac{\sigma_{zz}(z|\omega,\omega')}{\sigma(z|\omega,\omega')} = \zeta_z(z|\omega,\omega') + [\zeta(z|\omega,\omega')]^2, \quad (2a)$$

so that (see for instance appendix A of [5] or chapter XX of [6])

$$\sigma(-z) = -\sigma(z), \qquad \zeta(-z) = -\zeta(z), \qquad \gamma(-z) = \gamma(z), \quad (2b)$$
$$\sigma(0) = \gamma(0) = 0, \qquad \lim_{z\to 0}[z\zeta(z)] = 1, \quad (2c)$$
$$\sigma(z|\infty, \mathrm{i}\infty) = z, \qquad \zeta(z|\infty, \mathrm{i}\infty) = \frac{1}{z}, \qquad \gamma(z|\infty, \mathrm{i}\infty) = 0, \quad (2d)$$
$$\sigma\left(z\left|\frac{\pi}{2a}, \mathrm{i}\infty\right.\right) = \frac{\sinh(az)}{a}\exp\left(-\frac{a^2z^2}{6}\right), \quad (2e)$$
$$\zeta\left(z\left|\frac{\pi}{2a}, \mathrm{i}\infty\right.\right) = a\cot(az) + \frac{a^2z}{3}, \quad (2f)$$
$$\gamma\left(z\left|\frac{\pi}{2a}, \mathrm{i}\infty\right.\right) = \frac{a^2}{9}[a^2z^2 + 6az\cot(az) - 6], \quad (2g)$$
$$\zeta(x)\zeta(y) = -[\zeta(x) - \zeta(y)]\zeta(x-y) + \frac{1}{2}[\gamma(x) + \gamma(y) + \gamma(x-y)]. \quad (2h)$$

Above and throughout i denotes of course the imaginary unit. The two numbers ω, ω' (the dependence on which often is not displayed, as already done above) are the semiperiods generally associated with elliptic functions; they generate a tessellation of the complex ψ-plane, and we generally consider our dependent variables ψ_n within one parallelogram of this standard tessellation: the function $\psi_n(\vec{r}, t)$ is then defined in the rest of the complex ψ_n-plane via the standard double-periodicity properties associated with functions of elliptic type.

This system of nonlinear PDEs in multidimensional space is *solvable* provided the N dependent variables $\psi_n(\vec{r}, t)$ satisfy the constraint

$$\sum_{n=1}^{N}\psi_n(\vec{r}, t) = P, \quad (3)$$

with P a (time- and space-independent!) *constant*. The consistency of this condition with the system of N PDEs (1*a*) satisfied by the N dependent variables $\psi_n(\vec{r}, t)$ is an immediate consequence of the fact that, by summing the N PDEs (1*a*) over n from 1 to N, the double sum over the indices m and n on the right-hand side vanishes due to the antisymmetry of the summand under the exchange of these two indices (see (2*b*)).

Note incidentally that the specification $m \neq n$ in the sum on the right-hand side of (1*b*) is superfluous, see (2*c*). Also note that the PDE (1*b*) could be *linearized*—as regards the dependent variable $\theta(\vec{r}, t)$ it features—via the change of dependent variable $\Theta(\vec{r}, t) = \exp\left(-\frac{\theta(\vec{r},t)}{2}\right)$.

Particularly interesting—especially in the physical space with $S = 3$—are the *rotation-invariant* nonlinear 'Schrödinger' case characterized by $\alpha = i$ and $\beta = 0$, the *rotation-invariant* nonlinear 'diffusion' equation characterized by $\alpha = 1$ and $\beta = 0$ and the *relativistically invariant* nonlinear 'Klein–Gordon' case characterized by $\alpha = 0, \beta = 1$ and $\eta = -1$.

In the following section (and in the appendix), the *solvability* of this system of PDEs is demonstrated. In section 3, we show that the following two (systems of) PDEs,

$$\alpha\left[\tilde{\psi}_{n,t} - \frac{\mathrm{i}\Omega}{2}\vec{r}\cdot\vec{\nabla}\tilde{\psi}_n\right] + \eta\Delta\tilde{\psi}_n = \eta(\vec{\nabla}\tilde{\theta})\cdot(\vec{\nabla}\tilde{\psi}_n) + 2\eta\sum_{m=1,m\neq n}^{N}\zeta(\tilde{\psi}_n - \tilde{\psi}_m)(\vec{\nabla}\tilde{\psi}_n)\cdot(\vec{\nabla}\tilde{\psi}_m), \tag{4a}$$

$$\alpha\left[\tilde{\theta}_t - \frac{\mathrm{i}\Omega}{2}\vec{r}\cdot\vec{\nabla}\tilde{\theta}\right] + \eta\left[\Delta\tilde{\theta} - \frac{1}{2}(\vec{\nabla}\tilde{\theta})\cdot(\vec{\nabla}\tilde{\theta})\right] = -2\eta\sum_{m,n=1}^{N}\gamma(\tilde{\psi}_n - \tilde{\psi}_m)(\vec{\nabla}\tilde{\psi}_n)\cdot(\vec{\nabla}\tilde{\psi}_m), \tag{4b}$$

respectively

$$\begin{aligned}\beta\{\tilde{\psi}_{n,tt} &- \mathrm{i}\Omega[1 + 2(\vec{r}\cdot\vec{\nabla})]\tilde{\psi}_{n,t} - \Omega^2[(\vec{r}\cdot\vec{\nabla}) + (\vec{r}\cdot\vec{\nabla})(\vec{r}\cdot\vec{\nabla})]\tilde{\psi}_n\} + \eta\Delta\tilde{\psi}_n \\ &= \beta[\tilde{\theta}_t - \mathrm{i}\Omega(\vec{r}\cdot\vec{\nabla})\tilde{\theta}][\tilde{\psi}_{n,t} - \mathrm{i}\Omega(\vec{r}\cdot\vec{\nabla})\tilde{\psi}_n] + \eta(\vec{\nabla}\tilde{\theta})\cdot(\vec{\nabla}\tilde{\psi}_n) \\ &\quad + 2\sum_{m=1,m\neq n}^{N}\zeta(\tilde{\psi}_n - \tilde{\psi}_m)\{\eta(\vec{\nabla}\tilde{\psi}_n)\cdot(\vec{\nabla}\tilde{\psi}_m) \\ &\quad + \beta[\tilde{\psi}_{n,t} - \mathrm{i}\Omega(\vec{r}\cdot\vec{\nabla})\tilde{\psi}_n][\tilde{\psi}_{m,t} - \mathrm{i}\Omega(\vec{r}\cdot\vec{\nabla})\tilde{\psi}_m]\},\end{aligned} \tag{5a}$$

$$\begin{aligned}\beta\{\tilde{\theta}_{tt} &- \mathrm{i}\Omega[1 + 2(\vec{r}\cdot\vec{\nabla})]\tilde{\theta}_t - \Omega^2[(\vec{r}\cdot\vec{\nabla}) + (\vec{r}\cdot\vec{\nabla})(\vec{r}\cdot\vec{\nabla})]\tilde{\theta} \\ &\quad - [\tilde{\theta}_t - \mathrm{i}\Omega(\vec{r}\cdot\vec{\nabla})\tilde{\theta}]^2\} + \eta\left[\Delta\tilde{\theta} - \frac{1}{2}(\vec{\nabla}\tilde{\theta})\cdot(\vec{\nabla}\tilde{\theta})\right] \\ &= -2\sum_{m,n=1}^{N}\gamma(\tilde{\psi}_n - \tilde{\psi}_m)\{\eta(\vec{\nabla}\tilde{\psi}_n)\cdot(\vec{\nabla}\tilde{\psi}_m) \\ &\quad + \beta[\tilde{\psi}_{n,t} - \mathrm{i}\Omega(\vec{r}\cdot\vec{\nabla})\tilde{\psi}_n][\tilde{\psi}_{m,t} - \mathrm{i}\Omega(\vec{r}\cdot\vec{\nabla})\tilde{\psi}_m]\},\end{aligned} \tag{5b}$$

where Ω is a *positive* constant, are *isochronous*, namely [7] they possess many *isochronous* solutions characterized by the periodicity properties $\tilde{\psi}_n(\vec{r}, t + 2T) = \tilde{\psi}_n(\vec{r}, t)$ respectively $\tilde{\psi}_n(\vec{r}, t + T) = \tilde{\psi}_n(\vec{r}, t)$ with $T = 2\pi/\Omega$. In section 4, the relations of these *solvable* models involving elliptic functions to the analogous models involving trigonometric [3] or rational [1] functions are tersely discussed.

2. Results

The starting point of our treatment is the *linear* PDE

$$f\Psi + \alpha\Psi_t + \beta\Psi_{tt} + \eta\Delta\Psi = 0 \tag{6}$$

satisfied by the dependent variable $\Psi \equiv \Psi(\psi; \vec{r}, t)$. Here, subscripted variables denote again partial differentiations, for instance $\Psi_t \equiv \partial\Psi/\partial t$, and the quantities f, α, β, η are the *arbitrary* constants (for simplicity: they might actually depend on the time and space variables without affecting our main conclusion about the *solvable* character of the system of PDEs (1), see below). Note that the assumed dependence of the dependent variable $\Psi(\psi; \vec{r}, t)$ on the independent variable ψ is purely parametric, since this PDE entails no differentiations with respect to this variable—which for our purposes is a complex number inside the relevant parallelogram of the tessellation of the complex ψ-plane characterizing the double periodicity of elliptic functions associated with the two semiperiods ω and ω', see above and below.

We moreover focus hereafter on the special solution of this PDE (6) whose ψ-dependence is characterized by the *ansatz*

$$\Psi(\psi; \vec{r}, t) = \exp\left[-\frac{\theta(\vec{r}, t)}{2}\right]\prod_{n=1}^{N}\frac{\sigma[\psi - \psi_n(\vec{r}, t)]}{\sigma(\psi - p_n)}, \tag{7a}$$

$$\Psi(\psi; \vec{r}, t) = \varphi_0(\vec{r}, t) + \sum_{m=1}^{N}\varphi_m(\vec{r}, t)\zeta(\psi - p_n), \tag{7b}$$

where the constants p_n are N, essentially arbitrary, numbers (independent of the time t and the space variable $\vec{r}$), located in the standard parallelogram of the tessellation of the complex ψ-plane on which we focus. This *ansatz*—as it is well known [6]—defines an elliptic, doubly periodic function of the variable ψ, characterized by the (*fixed*!) N poles p_n, the corresponding N residues $\varphi_m(\vec{r}, t)$ and the N zeros $\psi_n(\vec{r}, t)$, provided there holds the following two properties: the sum of the N residues vanishes,

$$\sum_{m=1}^{N}\varphi_m(\vec{r}, t) = 0, \tag{7c}$$

and the sum of the N zeros coincides with the sum of the N poles (see (3)),

$$\sum_{n=1}^{N}\psi_n(\vec{r}, t) = \sum_{n=1}^{N}p_n \equiv P. \tag{7d}$$

This we assume hereafter—and we show below that this assumption is indeed consistent with the PDE (6) satisfied by $\Psi(\psi; \vec{r}, t)$. Note that the last of these conditions, (7*d*), entails the formulae

$$\sum_{n=1}^{N}\psi_{n,t}(\vec{r}, t) = \sum_{n=1}^{N}\vec{\nabla}\psi_n(\vec{r}, t) = \sum_{n=1}^{N}\Delta\psi_n(\vec{r}, t) = 0. \tag{7e}$$

Let us emphasize that the simultaneous validity of the two formulae (7*a*) and (7*b*) entails—once the N numbers p_n are assigned—the possibility of computing (in principle, of course, not explicitly) the $N + 1$ quantities $\psi_n(\vec{r}, t)$ (of course, up to a permutation, see (7*a*)) and $\theta(\vec{r}, t)$ from $N + 1$ arbitrarily given quantities $\varphi_j(\vec{r}, t)$, $j = 0, 1, \ldots, N$ (assigned of course within the class characterized by the restriction (7*c*)), and likewise of computing $N + 1$ quantities $\varphi_j(\vec{r}, t)$ from $N + 1$ given quantities $\psi_n(\vec{r}, t)$ and $\theta(\vec{r}, t)$ (assigned of course within the class characterized by the restriction (7*d*)). The first of these two tasks amounts essentially to computing the *zeros* of a given (doubly periodic) elliptic function (of the complex variable ψ), and the second to computing the *residues* at its poles of a given (doubly periodic) elliptic function (again, of the complex variable ψ).

It is moreover plain that the PDE (6) satisfied by $\Psi(\psi; \vec{r}, t)$ entails, via (7*b*), that the $N + 1$ functions $\varphi_j(\vec{r}, t)$ satisfy the set of $N + 1$ *decoupled linear* PDEs

$$f\varphi_j + \alpha\varphi_{j,t} + \beta\varphi_{j,tt} + \eta\Delta\varphi_j = 0, \qquad j = 0, 1, \ldots, N. \tag{8}$$

The compatibility of this set of PDEs with the condition (7*c*) is thereby evident.

It is now fundamental to recognize that the system of (*nonlinear* and *coupled*) PDEs satisfied—as implied by (6) with (7*a*)—by the $N + 1$ dependent variables $\psi_n(\vec{r}, t)$ and $\theta(\vec{r}, t)$ coincides with (1). The derivation of this result (via (2*h*) and (7*e*)) is in principle straightforward, yet we deem it appropriate to outline it tersely in the appendix.

Let us now indicate how the initial-value problem for the system (1) can be solved. To simplify the presentation we limit our treatment to the case with $\beta = 0$, when the initial data to be assigned are the $N + 1$ functions $\theta(\vec{r}, 0)$ and $\psi_n(\vec{r}, 0)$, the latter of which must of course satisfy the condition (3) (at $t = 0$): the main requirement is the $\vec{r}$-independence of the sum $\sum_{n=1}^{N} \psi_n(\vec{r}, 0)$, since we are free to assign the value of the constant P. The treatment of the case with $\beta \neq 0$ is quite analogous; the main difference is that in this case the initial data to be assigned also include the $N + 1$ functions $\psi_{n,t}(\vec{r}, 0)$ and $\theta_t(\vec{r}, 0)$. The details will be worked out without difficulty by the interested reader.

The solutions to be obtained are the $N + 1$ functions $\psi_n(\vec{r}, t)$ and $\theta(\vec{r}, t)$. The first step to obtain them is to compute, from the initial data $\psi_n(\vec{r}, 0)$ and $\theta(\vec{r}, 0)$, the $N + 1$ initial data $\varphi_j(\vec{r}, 0)$, $j = 0, 1, \ldots, N$, by identifying the right-hand sides of (7*a*) and (7*b*) at the initial time $t = 0$.

The second step is to compute the $N + 1$ functions $\varphi_j(\vec{r}, t)$ by integrating over time the (*decoupled*) set of $N + 1$ *linear* PDEs (8).

The third and last step is to compute—again, by identifying the right-hand sides of (7*a*) and (7*b*), but now at time t—the $N + 1$ functions $\psi_n(\vec{r}, t)$ and $\theta(\vec{r}, t)$ from the $N + 1$ functions $\varphi_j(\vec{r}, t)$.

3. Isochronous versions

Consider the system (1) with $\beta = f = 0$ (and with the independent variables $\vec{r}$ and t formally replaced by $\vec{\rho}$ and τ) and use the change of (independent) variables (the so-called trick [7])

$$\tilde{\psi}_n(\vec{r}, t) = \psi_n(\vec{\rho}, \tau), \qquad \tilde{\theta}(\vec{r}, t) = \theta(\vec{\rho}, \tau), \tag{9a}$$

with

$$\vec{\rho} \equiv \vec{\rho}(t) = \exp\left(\frac{\mathrm{i}\Omega t}{2}\right)\vec{r}, \qquad \tau \equiv \tau(t) = \frac{\exp(\mathrm{i}\Omega t) - 1}{\mathrm{i}\Omega}. \tag{9b}$$

It is then a matter of trivial differentiations to verify that the functions $\tilde{\psi}_n(\vec{r}, t)$ satisfy the system of PDEs (4).

A completely analogous treatment yields the system (5), except that one must now start with the system (1) with $\alpha = f = 0$ and use the change of variables (9) but with (9*b*) replaced by

$$\vec{\rho} \equiv \vec{\rho}(t) = \exp(\mathrm{i}\Omega t)\vec{r}, \qquad \tau \equiv \tau(t) = \frac{\exp(\mathrm{i}\Omega t) - 1}{\mathrm{i}\Omega}. \tag{9c}$$

These relations, (9), among the functions $\tilde{\psi}_n(\vec{r}, t)$, $\tilde{\theta}(\vec{r}, t)$ and $\psi_n(\vec{\rho}, \tau)$, $\theta(\vec{\rho}, \tau)$, together with the *solvable* character of the system (1), justify [7] the assertions made in the introductory section about the *isochronous* characters of the systems of PDEs (4) and (5).

4. Reductions

For $\omega = \infty$ and $\omega' = \mathrm{i}\infty$, the elliptic-type functions become very simple, in particular (see (2*d*)) $\zeta(z) = 1/z$ and $\gamma(z) = 0$. Provided one moreover sets $f = 0$, it is then consistent with (1*b*) to set $\theta(\vec{r}, t) = 0$, and (1*a*) then reads

$$\alpha\psi_{n,t} + \beta\psi_{n,tt} + \eta\Delta\psi_n = 2\sum_{m=1,m\neq n}^{N} \frac{\beta\psi_{n,t}\psi_{m,t} + \eta(\vec{\nabla}\psi_n)\cdot(\vec{\nabla}\psi_m)}{\psi_n - \psi_m}. \tag{10}$$

Hence, for $\alpha = i, \beta = 0, \eta = 1$ respectively $\alpha = 0, \beta = 1, \eta = -1$, this system of nonlinear evolution PDEs reduces to (a special case of) the models treated in [1].

For $\omega = \infty$ and $\omega' = \frac{\mathrm{i}\pi}{2a}$, the elliptic-type functions also simplify, albeit less drastically, see (2*e*), (2*f*) and (2*g*), hence the system of PDEs (1) reads

$$\begin{aligned}\alpha\psi_{n,t} + \beta\psi_{n,tt} + \eta\Delta\psi_n &= \beta\theta_t\psi_{n,t} + \eta(\vec{\nabla}\theta)\cdot(\vec{\nabla}\psi_n)\\ &\quad + 2\sum_{m=1,m\neq n}^{N}\left\{a\cot[a(\psi_n - \psi_m)] + \frac{a^2(\psi_n - \psi_m)}{3}\right\}\\ &\quad\cdot[\beta\psi_{n,t}\psi_{m,t} + \eta(\vec{\nabla}\psi_n)\cdot(\vec{\nabla}\psi_m)],\end{aligned} \tag{11a}$$

$$\begin{aligned}&\alpha\theta_t + \beta\left(\theta_{tt} - \frac{1}{2}\theta_t^2\right) + \eta\left[\Delta\theta - \frac{1}{2}(\vec{\nabla}\theta)\cdot(\vec{\nabla}\theta)\right]\\ &\quad = 2f - \frac{2a^2}{9}\sum_{m,n=1}^{N}\{a^2(\psi_n - \psi_m)^2 + 6a(\psi_n - \psi_m)\cot[a(\psi_n - \psi_m)]\}\\ &\quad\cdot[\beta\psi_{n,t}\psi_{m,t} + \eta(\vec{\nabla}\psi_n)\cdot(\vec{\nabla}\psi_m)].\end{aligned} \tag{11b}$$

To eliminate a term on the right-hand side of the last equation we used (7*e*). Using this same property the term $(\psi_n - \psi_m)^2$ appearing inside the double sum on the right-hand side of this last evolution equation could be replaced by $(-2\psi_n + \psi_m)\psi_m$; likewise, the term $a^2(\psi_n - \psi_m)/3$ appearing inside the sum on the right-hand side of the first of these two evolution equations could be replaced by $-a^2\psi_m/3$, at the cost of adding to the right-hand side (outside of the sum) the extra term $-2a^2\psi_n\big[\beta\psi_{n,t}^2 + \eta(\vec{\nabla}\psi_n)^2\big]\big/3$. Note that this system, (11), does not quite reduce to that treated in our previous paper [3], consistently with the fact that, via (2*e*), the *ansatz* (7) does not quite coincide with the *ansatz* (5) of [3].

Appendix. Derivation of the system of PDEs (1)

Logarithmic time differentiation of (7*a*) yields

$$\Psi_t(\psi;\vec{r},t) = -\Psi(\psi;\vec{r},t)\left\{\frac{1}{2}\theta_t(\vec{r},t) + \sum_{n=1}^{N}\zeta[\psi - \psi_n(\vec{r},t)]\psi_{n,t}(\vec{r},t)\right\}, \tag{A.1}$$

and an additional differentiation yields

$$\begin{aligned}\Psi_{tt} = \Psi\Bigg\{&-\frac{1}{2}\theta_{tt} - \sum_{n=1}^{N}\zeta(\psi - \psi_n)\psi_{n,tt} + \sum_{n=1}^{N}\zeta'(\psi - \psi_n)\psi_{n,t}^2\\ &+\left[\frac{1}{2}\theta_t + \sum_{n=1}^{N}\zeta(\psi - \psi_n)\psi_{n,t}\right]^2\Bigg\}\end{aligned} \tag{A.2a}$$

$$= \Psi \left\{ -\frac{1}{2}\theta_{tt} + \frac{1}{4}\theta_t^2 + \sum_{n=1}^{N} \zeta(\psi - \psi_n)[-\psi_{n,tt} + \theta_t \psi_{n,t}] \right.$$
$$\left. + \sum_{n=1}^{N} \gamma(\psi - \psi_n)\psi_{n,t}^2 + \sum_{m,n=1;m\neq n}^{N} \zeta(\psi - \psi_n)\zeta(\psi - \psi_m)\psi_{n,t}\psi_{m,t} \right\}. \qquad \text{(A.2}b\text{)}$$

To perform the second step we used (2a). We now use the identity (2h) and then the relevant relation (7e) and obtain thereby the formula

$$\Psi_{tt} = \Psi \left\{ -\frac{1}{2}\left[\theta_{tt} - \frac{1}{2}\theta_t^2 - \sum_{m,n=1}^{N} \gamma(\psi_n - \psi_m)\psi_{n,t}\psi_{m,t} \right] \right.$$
$$\left. + \sum_{n=1}^{N} \zeta(\psi - \psi_n) \left[-\psi_{n,tt} + \theta_t \psi_{n,t} + 2 \sum_{m=1,m\neq n}^{N} \zeta(\psi_n - \psi_m)\psi_{n,t}\psi_{m,t} \right] \right\}. \qquad \text{(A.2}c\text{)}$$

In the double sum on the right-hand side the restriction $m \neq n$ has been dropped thanks to the vanishing of $\gamma(z)$ at $z = 0$, see (2c).

Likewise one obtains the formulae

$$\vec{\nabla}\Psi(\psi; \vec{r}, t) = -\Psi(\psi; \vec{r}, t) \left\{ \frac{1}{2}\vec{\nabla}\theta(\vec{r}, t) + \sum_{n=1}^{N} \zeta[\psi - \psi_n(\vec{r}, t)]\vec{\nabla}\psi_n(\vec{r}, t) \right\}, \qquad \text{(A.3)}$$

$$\Delta\Psi = \Psi \left\{ -\frac{1}{2}\left[\Delta\theta - \frac{1}{2}(\vec{\nabla}\theta)\cdot(\vec{\nabla}\theta) - \sum_{m,n=1}^{N} \gamma(\psi_n - \psi_m)(\vec{\nabla}\psi_n)\cdot(\vec{\nabla}\psi_m) \right] \right.$$
$$+ \sum_{n=1}^{N} \zeta(\psi - \psi_n) \left[-\Delta\psi_{n,tt} + (\vec{\nabla}\theta)\cdot(\vec{\nabla}\psi_n) \right.$$
$$\left.\left. + 2 \sum_{m=1,m\neq n}^{N} \zeta(\psi_n - \psi_m)(\vec{\nabla}\psi_n)\cdot(\vec{\nabla}\psi_m) \right] \right\}. \qquad \text{(A.4)}$$

The insertion in (6) of these formulae, (A.1), (A.2c), (A.4), yields (1).

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